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ANOTHER COMPARISON THEOREM FOR

DIFFERENTIAL GAMES

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February 20, 1973

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ABSTRACT

A result is given for comparing the values of two fixed duration differential games whose dynamics are described by possibly different numbers of differential equations over possibly different time intervals. Roughly speaking, proof of the result makes use of conditions which allow a certain transformation of the trajectories of one game into those of the other game to be accomplished. An application is given for a certain class of differential games, and an example.

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1. Introduction. In order to ascertain the value of a differential game it sometimes is fruitful to compare it to another differential game with simpler structure, and thereby obtain an inequality of the values. In [4] Rao obtained a result of such a type, by making use of certain correspondences between the two games of interest, and the theory of differential inequalities. In [2] Friedman derived a value inequality which depended on an inequality of Hamiltonian functions and certain results of parabolic partial differential equations. Stern in [5] gave bounds on the value of a certain type of survival game by transforming it into a game of fixed duration.

In the next section a brief review of the required concepts of differential game theory, as in [3], is given. In section 3 the main result is presented. The result, in essence, is an inequality of the values of two games with path-functional payoffs when a certain mapping of the trajectories of one game into those of the other can be accomplished. In section 4 certain generalizations of the result are stated. Then an application of the main result to a specific class of games and an example are given.

2. Preliminaries. Consider a system of m ordinary differential equations

(2.1) $$\dot{x} = f(t,x,y,z) \qquad (t_0 \leq t \leq T_0)$$

with an initial condition

(2.2) $$x(t_0) = x_0.$$

Denote by Y and Z compact subsets of the Euclidean spaces R^n and R^q respectively. The controls $y(t)$ and $z(t)$ are Lebesgue measurable functions taking values almost everywhere in Y and Z respectively, defined on $[t_0,T_0]$.

We shall consider a payoff of the form

$$P(y,z) = g\big(x(T_0)\big) + \int_{t_0}^{T_0} h(t,x)dt. \tag{2.3}$$

A set of assumptions which guarantee a unique solution of (2.1)-(2.2) for any pair of controls are the following (see [1] and [3]):

(a) $f(t,x,y,z)$ is continuous on $[t_0,T_0] \times R^m \times Y \times Z$.

(b) There exists $k(t) \in L^1(t_0,T_0)$ such that

$$|f(t,x,y,z)| \leq k(t)(1 + |x|)$$

for all $(t,x,y,z) \in [t_0,T_0] \times R^m \times Y \times Z$.

(c) For each $R > 0$ there exists $k_R(t) \in L^1(t_0,T_0)$ such that

$$|f(t,x,y,z) - f(t,\bar{x},y,z)| \leq k_R(t)|x - \bar{x}|$$

for all $t \in [t_0,T_0]$, $y \in Y$, $z \in Z$, $|x| \leq R$ and $|\bar{x}| \leq R$.

Concerning the payoff we shall assume

(d) $h(t,x)$ is continuous on $[t_0,T_0] \times R^m$.

(e) g is continuous on R^m.

The above assumptions guarantee that the payoff functional $P(y,z)$ is well defined.

Let n be a positive integer nd $\delta = \dfrac{T_0 - t_0}{n}$. Denote

$$I_j = (t_{j-1},t_j) \quad \text{for } t_j = t_0 + j\delta, \qquad 1 \leq j \leq n.$$

Define Y_j and Z_j to be the classes of measurable functions on I_j which almost everywhere take values in Y and Z respectively.

Let $\Gamma^{\delta,j}$ be any map of $Z_1 \times Y_1 \times Z_2 \times Y_2 \times \cdots \times Y_{j-1} \times Z_j$ into Y_j. We then call the n-tuple

$$\Gamma^\delta = (\Gamma^{\delta,1},\ldots,\Gamma^{\delta,n})$$

an <u>upper</u> δ-<u>strategy</u> for y. Similarly, we define an upper δ-strategy for z, Δ^δ, whose components $\Delta^{\delta,j}$ are maps from $Y_1 \times Z_1 \times Y_2 \times Z_2 \times \cdots \times Z_{j-1} \times Y_j$ into Z_j.

For $(2 \leq j \leq n)$ let $\Gamma_{\delta,j}$ be any map from $Y_1 \times Z_1 \times Y_2 \times Z_2 \times \cdots \times Y_{j-1} \times Z_{j-1}$ into Y_j, and let $\Gamma_{\delta,1}$ be any element of Y_1. We then call the n-tuple

$$\Gamma_\delta = (\Gamma_{\delta,1}, \ldots, \Gamma_{\delta,n})$$

a lower δ-strategy for y. Analogously one defines a lower δ-strategy Δ_δ for z.

Given a pair $(\Delta_\delta, \Gamma^\delta)$ we uniquely obtain control functions (z_δ, y^δ), and a trajectory x^δ. (z_δ, y^δ) is called the outcome of $(\Delta_\delta, \Gamma^\delta)$. Analogously, a pair $(\Gamma_\delta, \Delta^\delta)$ yields an outcome (y_δ, z^δ) and a trajectory x_δ.

The upper δ-value is the number

$$V^\delta = \inf_{\Delta_{\delta,1}} \sup_{\Gamma^{\delta,1}} \inf_{\Delta_{\delta,2}} \sup_{\Gamma^{\delta,2}} \ldots \inf_{\Delta_{\delta,n}} \sup_{\Gamma^{\delta,n}} P[\Delta_\delta, \Gamma^\delta],$$

and the lower δ-value is defined by

$$V_\delta = \sup_{\Gamma_{\delta,1}} \inf_{\Delta^{\delta,1}} \sup_{\Gamma_{\delta,2}} \inf_{\Delta^{\delta,2}} \ldots \sup_{\Gamma_{\delta,n}} \inf_{\Delta^{\delta,n}} P[\Gamma_\delta, \Delta^\delta].$$

We say the differential game has value V if the limits $\lim_{\delta\to 0} V_\delta = V^-$ and $\lim_{\delta\to 0} V^\delta = V^+$ exist and are equal. V^- and V^+ are called the lower and upper values respectively.

A result, proven in [3], which we will make use of is the following:

Theorem 2.1. Let (a)-(e) hold. Then

(2.4) $$V^\delta = \inf_{\Delta_\delta} \sup_{\Gamma^\delta} P[\Delta_\delta, \Gamma^\delta] = \sup_{\Gamma^\delta} \inf_{\Delta_\delta} P[\Delta_\delta, \Gamma^\delta],$$

and

(2.5) $$V_\delta = \sup_{\Gamma_\delta} \inf_{\Delta^\delta} P[\Gamma_\delta, \Delta^\delta] = \inf_{\Delta^\delta} \sup_{\Gamma_\delta} P[\Gamma_\delta, \Delta^\delta].$$

We now state an assumption on the form of the dynamics

(f) $f(t,x,y,z) = f_1(t,x,y) + f_2(t,x,z)$.

In [3] the following theorem's proof may be found.

Theorem 2.2. Let (a)-(e) *hold. Then*

(i) V^+ *and* V^- *exist.*

(ii) *If also* (f) *holds then* $V^+ = V^- = V$.

3. Main results. In this section we will derive a theorem for comparing the values of two differential games G and $\bar{G}$. The game G has for its dynamics a system of m differential equations given by

$$\dot{x} = f_1(t,x,y) + f_2(t,x,z) \qquad (t_0 \leq t \leq T_0) \tag{3.1}$$

and an initial condition

$$x(t_0) = x_0. \tag{3.2}$$

The control sets Y and Z are subsets of R^n and R^q respectively, and the payoff is given by $P(y,z)$ as in expression (2.3).

The game $\bar{G}$ has its dynamics described by a system of $\bar{m}$ differential equations on an interval $[\bar{t}_0,\bar{T}_0]$

$$\dot{\bar{x}} = \bar{f}_1(\bar{t},\bar{x},\bar{y}) + \bar{f}_2(\bar{t},\bar{x},\bar{y}) \qquad (\bar{t}_0 \leq \bar{t} \leq \bar{T}_0) \tag{3.3}$$

with initial condition

$$\bar{x}(\bar{t}_0) = \bar{x}_0. \tag{3.4}$$

The control sets $\bar{Y}$ and $\bar{Z}$ are subsets of Euclidean spaces $R^{\bar{n}}$ and $R^{\bar{q}}$ respectively. The payoff for $\bar{G}$ is given by

$$\bar{P}(\bar{y},\bar{z}) = \bar{g}\big(\bar{x}(\bar{T}_0)\big) + \int_{\bar{t}_0}^{\bar{T}_0} \bar{h}(\bar{t},\bar{x})d\bar{t}. \tag{3.5}$$

Let R_0 denote a bound in the sup-norm on the trajectories of game G; that is, $|x(t)| \leq R_0$ for all $t \in [t_0,T_0]$ for any possible path $x(t)$, where $|\ |$ denotes the Euclidean norm. For game $\bar{G}$ a number $\bar{R}_0$ is similarly defined.

Let $q:[\bar{t}_0,\bar{T}_0] \to [t_0,T_0]$ be continuously differentiable function whose derivative satisfies $q'(\bar{t}) > 0$ on $[\bar{t}_0,\bar{T}_0]$.

$\mathfrak{M}$ denotes an $(\bar{m} \times m)$ constant matrix. (**Generalizations follow in section 4.**)

The following elementary lemma will be useful.

Lemma 3.1. *Let (a)-(c) hold and assume that* $\bar{x}_0 = \mathfrak{M} x_0$. *Let* $\{y(t), z(t), x(t)\}$ *and* $\{\bar{y}(\bar{t}), \bar{z}(\bar{t}), \bar{x}(\bar{t})\}$ *be triples of controls and corresponding paths for games* G *and* $\bar{G}$ *respectively. If*

$$\bar{f}_1(\bar{t}, \bar{x}(\bar{t}), \bar{y}(\bar{t})) = q'(\bar{t}) \mathfrak{M} f_1\big(q(\bar{t}), x(q(\bar{t})), y(q(\bar{t}))\big) \tag{3.6}$$

and

$$\bar{f}_2(\bar{t}, \bar{x}(\bar{t}), \bar{z}(\bar{t})) = q'(\bar{t}) \mathfrak{M} f_2\big(q(\bar{t}), x(q(\bar{t})), z(q(\bar{t}))\big) \tag{3.7}$$

then it follows that

$$\bar{x}(\bar{t}) = \mathfrak{M} x(q(\bar{t})) \qquad (\bar{t}_0 \le \bar{t} \le \bar{T}_0). \tag{3.8}$$

Proof. By the chain-rule we have

$$\frac{d}{d\bar{t}} \mathfrak{M} x(q(\bar{t})) = q'(\bar{t}) \mathfrak{M} \Big[f_1\big(q(\bar{t}), x(q(\bar{t})), y(q(\bar{t}))\big) + f_2\big(q(\bar{t}), x(q(\bar{t})), z(q(\bar{t}))\big)\Big]. \tag{3.9}$$

Upon making the identifications (3.6) and (3.7), (3.8) easily follows.

Now we introduce the following condition:

(L) $f_1(f_2)$ is uniformly Lipschitz continuous in x on $[t_0, T_0] \times \{|x| \le R_0\} \times Y$ $([t_0, T_0] \times \{|x| \le R_0\} \times Z)$ and $\bar{f}_1(\bar{f}_2)$ is uniformly Lipschitz continuous in $\bar{x}$ on $[\bar{t}_0, \bar{T}_0] \times \{|\bar{x}| \le \bar{R}_0\} \times \bar{Y}$ $([\bar{t}_0, \bar{T}_0] \times \{|\bar{x}| \le \bar{R}_0 \times \bar{Z})$.

The following conditions will be required:

(C_1) Given $\bar{t} \in [\bar{t}_0, \bar{T}_0]$, $\bar{x} \in R^{\bar{m}}$ such that $|\bar{x}| \le \bar{R}_0$, $x \in R^m$ such that $|x| \le R_0$, and $y \in Y$, there exists $\bar{y} \in \bar{Y}$ such that $\bar{f}_1(\bar{t}, \bar{x}, \bar{y}) = q'(\bar{t}) \mathfrak{M} f_1(q(\bar{t}), x, y)$.

(C_2) Given $\bar{t} \in [\bar{t}_0, \bar{T}_0]$, $\bar{x} \in R^m$ such that $|\bar{x}| \le \bar{R}_0$, $x \in R^m$ such that $|x| \le R_0$, and $\bar{z} \in \bar{Z}$, there exists $z \in Z$ such that

$\bar{f}_2(\bar{t},\bar{x},\bar{z}) = q'(\bar{t})\,\mathfrak{M} f_2(q(\bar{t}),x,z).$

(C_3) $g(x(T_0)) \leq \bar{g}(\bar{x}(\bar{T}_0))$ for all attainable end-states $x(T_0)$ and $\bar{x}(\bar{T}_0)$ for G and $\bar{G}$ respectively.

(C_4) $\int_{t_0}^{T_0} h(t,x(t))dt \leq \int_{\bar{t}_0}^{\bar{T}_0} \bar{h}(\bar{t},\mathfrak{M}x(q(\bar{t})))d\bar{t}$ for all paths $x(t)$ of game G.

Our main result is the following:

__Theorem 3.1__. __Let the games__ G __and__ $\bar{G}$ __satisfy__ (a)-(f). __Assume__ $\bar{x}_0 = \mathfrak{M}x_0$ __and let__ (L), (C_1)-(C_4) __hold__. __Then__

(3.10) $$V(G) \leq V(\bar{G}),$$

__where__ V(G) __denotes the value of__ G __and__ $V(\bar{G})$ __that of__ $\bar{G}$.

__Proof__. Let n be a positive integer and $\bar{\delta} = \frac{\bar{T}_0 - \bar{t}_0}{n}$. Let $\bar{I}_j = (\bar{t}_{j-1},\bar{t}_j)$ as in section 2, $j = 1,2,\ldots,n$. Also define intervals $I_j = (t_{j-1},t_j) = (q(\bar{t}_{j-1}),q(\bar{t}_j))$, $j = 1,2,\ldots,n$. Call the partition corresponding to the latter subdivision π, and denote by $|\pi|$ its mesh. Analogously to δ-strategies, one can in the obvious way define π-strategies. Analogously to $V^-(G)$ and $V^+(G)$, numbers $V_\pi(G)$ and $V^\pi(G)$ can be defined. Also, under the assumptions of Theorem 2.2, $V_\pi(G)$ and $V^\pi(G)$ converge to V(G) for any sequence of partitions $\{\pi\}$ such that $|\pi| \to 0$. (See [3], p. 32.)

Let Γ_π be an arbitrary lower π-strategy for player y in game G, and let $\Delta_{\bar{\delta}}$ be an arbitrary lower $\bar{\delta}$-strategy for player $\bar{z}$ in game $\bar{G}$. Here δ and π are understood to be related as above. Let $\varepsilon > 0$ be arbitrary. Suppose we can find controls $z(t)$ and $\bar{y}(\bar{t})$ such that

(3.11) $$P(\Gamma_\pi,z) \leq \bar{P}(\Delta_{\bar{\delta}},\bar{y}) + \varepsilon.$$

Upon letting $\delta \to 0$, Theorems 2.1, 2.2 and the remarks of the preceding

paragraph th n yield the desired result, since ε is arbitrary. The remainder of t e proof will be a construction of these controls. In view of $(C_3),(C_4)$, L mma 3.1, and the fact that $\bar{x}_0 = \mathfrak{M}x_0$, (3.11) will hold if $\bar{x}(\bar{t}) = \mathfrak{M}x\big(q(\ )\big)$ for all $\bar{t} \in [\bar{t}_0,\bar{T}_0]$ where $\bar{x}$ is the trajectory corresponding to $(\Delta_{\bar{\delta}},\bar{y})$ wh le x is the trajectory corresponding to (Γ_π,z). By assumption (d) it follo s that (3.11) will hold if, for prescribed $\gamma(\varepsilon) > 0$

$$(3.12) \qquad \left|\bar{x}(\bar{t}) - \mathfrak{M}x\big(q(\bar{t})\big)\right| \leq \gamma(\varepsilon) \quad \text{for all } t \in [\bar{t}_0,\bar{T}_0].$$

Subdivi e I_1 into intervals $S_k = [t_{k-1},t_k)$ such that $\bigcup_{k=1}^{p} S_k = I_1$. Make the rel ted subdivision $\bar{S}_k$ of the interval $\bar{I}_1$, using the correspondence $q:[\bar{t}_0,\bar{T}_0] \to \ t_0,T_0]$.

Γ_π dete mines a control $y_{\pi,1}(t)$ on I_1 and $\Delta_{\bar{\delta}}$ determines a control $z_{\bar{\delta}}(\bar{t})$ on $\bar{I}_1$. Using (C_1) we may choose a control $\bar{y}(\bar{t})$ on $\bar{S}_1$ such that

$$(3.13) \qquad {}_1\big(\bar{t},\bar{x}_0,\bar{y}(\bar{t})\big) = q'(\bar{t})\,\mathfrak{M}f_1\Big(q(\bar{t}),x_0,y_{\pi,1}\big(q(\bar{t})\big)\Big)$$

and by (C_2) z(t) on S_1 may be chosen so that

$$(3.14) \qquad {}_2\big(\bar{t},\bar{x}_0,z_{\bar{\delta},1}(\bar{t})\big) = q'(\bar{t})\,\mathfrak{M}f_2\Big(q(\bar{t}),x_0,z\big(q(\bar{t})\big)\Big).$$

Continuing i this way, choose $\bar{y}(\bar{t})$ on $\bar{S}_k$ and $z(t)$ on S_k, $k = 2,3,\ldots,p$, such that

$$(3.15) \qquad \bar{f}_1\ \bar{t},\bar{x}(\bar{t}_{k-1}),\bar{y}(\bar{t})\big) = q'(\bar{t})\,\mathfrak{M}f_1\Big(q(\bar{t}),x(t_{k-1}),y_{\pi,1}\big(q(\bar{t})\big)\Big)$$

and

$$(3.16) \qquad \bar{f}_2\ \bar{t},\bar{x}(t_{k-1}),z_{\bar{\delta},1}(\bar{t})\big) = q'(\bar{t})\,\mathfrak{M}f_2\Big(q(\bar{t}),x(t_{k-1}),z\big(q(\bar{t})\big)\Big).$$

There i a positive constant C such that

$$(3.17) \qquad |x(t_k) - x(t_{k-1})| \leq C \text{ meas. } S_k$$

and

$$(3.18) \qquad |\bar{x}(\bar{t}_k) - \bar{x}(\bar{t}_{k-1})| \leq C \text{ meas. } \bar{S}_k$$

for $k = 1,2,\ldots,p$. (See [3] for details of (3.17),(3.18).)

Using now the assumption (L), it follows that for some positive D, independent of partitions S_k and $\bar{S}_k$ we have

$$(3.19)\quad \left|\bar{f}_1(\bar{t},\bar{x}(\bar{t}),\bar{y}(\bar{t})) - q'(\bar{t})\mathfrak{M} f_1\big(q(\bar{t}),x(q(\bar{t})),y_{\pi,1}(q(\bar{t}))\big)\right| \leq D(\text{meas. } S_k + \text{meas. } \bar{S}_k)$$

and

$$(3.20)\quad \left|\bar{f}_2\big(\bar{t},\bar{x}(\bar{t}),z_{\bar{\delta},1}(\bar{t})\big) - q'(\bar{t})\mathfrak{M} f_2\big(q(\bar{t}),x(q(\bar{t})),z(q(\bar{t}))\big)\right| \leq D(\text{meas. } S_k + \text{meas. } \bar{S}_k)$$

for $\bar{t} \in \bar{S}_k$. Thus, by choosing sufficiently fine partitions S_k and $\bar{S}_k$ of I_1 and $\bar{I}_1$ respectively we can guarantee $|\bar{x}(\bar{t}) - \mathfrak{M}\, x(q(\bar{t}))| \leq \frac{\gamma(\varepsilon)}{n}$ holds on $\bar{I}_1$. The procedure is continued in the remaining intervals I_j and $\bar{I}_j$, $j = 2,3,\ldots,n$, and thus (3.12) follows.

4. <u>Generalizations and an example</u>. Certain extensions of Theorem 3.1 now follow.

(i) The condition (L) can be removed. This follows from the fact that if (L) does not hold, f_1, f_2, $\bar{f}_1$ and $\bar{f}_2$ can be uniformly approximated on their respective domains by functions which satisfy (L) as well as (a)-(c). By arguments in [2] and [3], if $\{f_1^n\}$, $\{f_2^n\}$, $\{\bar{f}_1^n\}$, $\{\bar{f}_2^n\}$ are sequences of functions satisfying (a),(b),(c) and (L) such that

$$\begin{aligned}
f_1^n &\to f_1 \quad \text{uniformly on } [t_0,T_0]\times\{|x| \leq R_0\}\times Y\times Z\\
f_2^n &\to f_2 \quad \text{uniformly on } [t_0,T_0]\times\{|x| \leq R_0\}\times Y\times Z\\
\bar{f}_1^n &\to \bar{f}_1 \quad \text{uniformly on } [\bar{t}_0,\bar{T}_0]\times\{|\bar{x}| \leq \bar{R}_0\}\times \bar{Y}\times \bar{Z}\\
\bar{f}_2^n &\to \bar{f}_2 \quad \text{uniformly on } [\bar{t}_0,\bar{T}_0]\times\{|\bar{x}| \leq \bar{R}_0\}\times \bar{Y}\times \bar{Z}
\end{aligned}$$

then

$$V^n(G) \to V(G)$$

and

$$V^n(\bar{G}) \to V(\bar{G})$$

where $V^n(G)$ is the value of the differential game corresponding to (3.2), (2.3) and dynamics $\dot{x} = f_1^n + f_2^n$, and where $V^n(\bar{G})$ is the value of the game corresponding to (3.4),(3.5) and dynamics $\dot{\bar{x}} = \bar{f}_1^n + \bar{f}_2^n$.

(ii) Suppose now that $\mathfrak{M} = \mathfrak{M}(\bar{t})$; in particular assume each entry in the matrix is a continuously differentiable function. Denote the Jacobian of $\mathfrak{M}(\bar{t})$ by $\mathfrak{M}'(\bar{t})$. Theorem (3.1) will still be true if we modify (C_1) and (C_2) as follows:

$(C_1)'$ Given $\bar{t} \in [\bar{t}_0, \bar{T}_0]$, $\bar{x} \in R^{\bar{m}}$ such that $|\bar{x}| \leq \bar{R}_0$, $x \in R^m$ such that $|x| \leq R_0$, and $y \in Y$, there exists $\bar{y} \in \bar{Y}$ such that

$$\bar{f}_1(\bar{t},\bar{x},\bar{y}) = q'(\bar{t})\mathfrak{M}(\bar{t})f_1\big(q(\bar{t}),x,y\big) + \mathfrak{M}'(\bar{t})x.$$

$(C_2)'$ Given $\bar{t} \in [\bar{t}_0, \bar{T}_0]$, $\bar{x} \in R^m$ such that $|\bar{x}| \leq \bar{R}_0$, $x \in R^m$ such that $|x| \leq R_0$, and $\bar{z} \in \bar{Z}$, there exists $z \in Z$ such that

$$\bar{f}_2(\bar{t},\bar{x},\bar{z}) = q'(\bar{t})\mathfrak{M}(\bar{t})f_2\big(q(\bar{t}),x,z\big) + \mathfrak{M}'(\bar{t})x.$$

(iii) Suppose (f) does not hold. If we replace (C_1) and (C_2) by the condition

$(C_{1,2})$ Given $\bar{t} \in [\bar{t}_0, \bar{T}_0]$, $\bar{x} \in R^{\bar{m}}$ such that $|\bar{x}| \leq R_0$, $x \in R^m$ such that $|x| \leq R_0$, $y \in Y$, and $\bar{z} \in \bar{Z}$, then there exist $\bar{y} \in \bar{Y}$ and $z \in Z$ such that

$$\bar{f}(\bar{t},\bar{x},\bar{y},\bar{z}) = q'(\bar{t})\mathfrak{M}(\bar{t})f\big(q(\bar{t}),x,y,z\big) + \mathfrak{M}'(t)x,$$

then the proof of Theorem 3.1 yields the following:

$$V^+(G) \leq V^+(\bar{G})$$

and

$$V^-(G) \leq V^-(\bar{G}).$$

(If the lower and upper Hamiltonians for G are equal, and also for $\bar{G}$, then the values exist even in the absence of condition (f) — see [2] for terminology.)

(iv) If $\mathcal{M}$ (or $\mathcal{M}(t)$ as in (ii) above) is a nonlinear but continuously differentiable mapping of R^m into $R^{\bar{m}}$ then assumptions (C_1), (C_2) and (C_4) can be modified in the obvious way so as to yield Theorem 3.1 also in this case.

We will now apply Theorem 3.1 to a specific class of differential games. Suppose that the game G with dynamics (3.3)-(3.4) has a payoff of the form

$$(4.1) \qquad P(y,z) = \int_{t_0}^{T_0} w(t)\cdot p(\mathcal{M}x(t))dt$$

where $w(t) > 0$. Let $q(\bar{t})$ denote a solution to the differential equation $w(q(\bar{t}))\cdot q'(\bar{t}) = 1$ on some interval $[\bar{t}_0, \bar{T}_0]$ subject to $q(\bar{t}_0) = t_0$ and $q(\bar{T}_0) = T_0$, if such a solution can be found. Consider now the payoff for comparison game $\bar{G}$ to be given by

$$(4.2) \quad \bar{P}(\bar{y},\bar{z}) = \int_{\bar{t}_0}^{\bar{T}_0} p(\bar{x}(\bar{t}))d\bar{t}$$

If (3.8) holds, then $P(y,z) = \bar{P}(\bar{y},\bar{z})$, as easily verified. Thus in order to apply Theorem 3.1, one needs to determine dynamics and control sets for $\bar{G}$ so as to satisfy conditions (C_1) and (C_2).

<u>example</u>. In the following example $m = 2$. G has dynaimics given by

$$\dot{x}_1 = ty^2 + tz$$
$$\dot{x}_2 = t^2y + 4tz$$
$$x(\sqrt{2}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and the payoff is given by

$$P(y,z) = \int_{\sqrt{2}}^{2} t(x_1(t) + x_2(t))^2 dt.$$

Y is the interval $[-1,1]$ and Z is the interval $[-2,1]$. For the comparison game $\bar{G}$ we take $\bar{P}(\bar{y},\bar{z}) = \int_1^2 \bar{x}^2 dt$ and dynamics

$$\dot{\bar{x}} = \bar{y} + \bar{z} \qquad ; \quad \bar{x}(1) = 3$$

Letting $\mathcal{M} = (1\ 1)$, $q(\bar{t}) = \sqrt{2\bar{t}}$, $Y = [3,3]$ and $Z = [0,5]$ we have $V(G) \leq V(\bar{G}) = 9$.

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